TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 174, December 1972

# $Q$-COMPOSABLE PROPERTIES, SEMIGROUPS AND CM-HOMOMORPHISMS

BY

A. R. BEDNAREK AND K. D. MAGILL, JR.(1)

ABSTRACT. A certain type of topological property is investigated. To each such property and each topological space satisfying various conditions there is associated, in a natural way, a semigroup of relations. The non-constant, union and symmetry preserving homomorphisms from one such semigroup into another are completely determined and this results in a topological version of the Clifford-Miller Theorem on endomorphisms of the full binary relation semigroup on a set.

The concept of $Q$-composable property was introduced in [3]. The study of such topological properties was motivated by the fact that for certain topological spaces, the family of all those binary relations on the space which have a $Q$-composable property is a semigroup under the operation of composition of relations. In [3], the isomorphisms between two such semigroups were completely determined. About the same time, A. H. Clifford and D. D. Miller introduced in [1] what we refer to here as a CM-homomorphism and in their main result, they completely determined the CM-endomorphism of the semigroup of *all* binary relations on a set. Then in [4], [5], the Clifford-Miller Theorem was given a topological setting when the semigroup under consideration was the semigroup of all closed relations on a particular type of topological space and the CM-homomorphisms from one such semigroup into another were completely determined. In this paper, we look at the analogous problem for semigroups of relations which have a $Q$-composable property.

In §1, $Q$-composable properties are defined and some of their more useful characteristics are determined. For example, it is shown that in numerous instances, such properties are closed under the operations of taking finite products, finite unions, finite intersections, closed subspaces and continuous images. Several of these facts play a crucial role in the proof of the homomorphism theorem. §2 is devoted to a further investigation of certain types of $Q$-composable properties and the homomorphism theorem and some related results are proven in

Received by the editors August 25, 1971.

*AMS (MOS) subject classifications* (1970). Primary 54A99, 20M20.

*Key words and phrases.* Topological properties, semigroups of relations.

(1) Most of the research for this paper was done while the second author was visiting the University of Florida. Their kind hospitality and financial support are gratefully acknowledged.

§3. Applications to specific $Q$-composable properties such as compactness are made in §4.

1. **Some results on $Q$-composable properties.** Since we intend to prove some things about topological properties, we begin by making this notion precise.

**Definition (1.1).** By a topological property $Q$, we mean a class of topological spaces such that if $X \in Q$ and $Y$ is homomorphic to $X$, then $Y \in Q$. When we say that $X$ has property $Q$, we mean simply that $X \in Q$.

**Remarks.** The property of compactness is the class of all compact spaces, *normality* is the class of all normal spaces, *connectedness* is the class of all connected spaces, and so on.

Before proceeding with some more definitions, we adopt the convention in this paper that the empty set can be regarded as a topological space and is a subspace of any other topological space. It has, of course, exactly one open subset.

**Definition (1.2).** A topological property $Q$ is said to be *pairwise productive* if $X \times Y$ has property $Q$ when both $X$ and $Y$ have property $Q$.

**Definition (1.3).** A topological property $Q$ is said to be *hereditary* if whenever $X$ has property $Q$, then all subspaces of $X$ have property $Q$. It is a *closed-hereditary* property if whenever $X$ has property $Q$, then each closed subset of $X$ has property $Q$.

**Proposition (1.4).** *Let $Q$ be any topological property which is pairwise productive and closed-hereditary. Suppose further that every space with property $Q$ is a Hausdorff space and that at least one space with property $Q$ has more than one point. Then the free union of any two spaces with property $Q$ has property $Q$.*

**Proof.** Suppose that $A$ and $B$ have property $Q$ and that $W$ is a space with more than one point which also has property $Q$. The conclusion is immediate if either $A$ or $B$ is empty so we assume they both are not. Choose $a \in A$, $b \in B$ and any two distinct points $w_1$ and $w_2$ in $W$. Since $Q$ is pairwise productive, $A \times B \times W$ has property $Q$ and, since the spaces involved are Hausdorff, one easily shows that

$$A \times \{b\} \times \{w_1\} \cup \{a\} \times B \times \{w_2\} \tag{1.4.1}$$

is a closed subset of $A \times B \times W$. Thus, since $Q$ is closed-hereditary (1.4.1) must also have property $Q$. This concludes the proof since the free union of $A$ and $B$ is homeomorphic to (1.4.1).

**Definition (1.5).** Let $Q$ be any topological property which is pairwise productive and hereditary and let $P$ be a topological property which is contained in $Q$. The property $P$ is said to be *$Q$-composable* if

(1.5.1) the two-point discrete space has property $P$, and

(1.5.2) for any space $X \in Q$ and any two subspaces $\alpha$ and $\beta$ of $X \times X$, $\alpha \circ \beta$ has property $P$ whenever both $\alpha$ and $\beta$ have property $P$ where $\alpha \circ \beta$ is given by

$$\alpha \circ \beta = \{(x, y) \in X \times X\colon (x, z) \in \beta \text{ and } (z, y) \in \alpha \text{ for some } z \in X\}.$$

The property $Q$ will be referred to as the *bounding* property for $P$.

We emphasize the fact that *any bounding property will be assumed to be pairwise productive and hereditary.* We will henceforth use the letter $H$ to denote the class of all Hausdorff spaces, the letters $CR$ to denote the class of all completely regular Hausdorff spaces, and the letter $T$ to denote the class of all topological spaces. All three classes are topological properties which are pairwise productive and hereditary and we will be primarily interested when our bounding properties are one of these.

The proofs of the next two results are omitted since they are minor modifications of the easy proofs of Proposition (2.1) and (2.2) of [3, p. 266]. In proving the first result, however, it is convenient to note that the bounding property $Q$ has spaces with more than one point. In fact, it follows from (1.5.1) that the two-point discrete space belongs to $Q$ and this, in turn, implies that all finite discrete spaces have property $Q$ since $Q$ is pairwise productive and hereditary.

**Proposition (1.6).** *Let $P$ be any $Q$-composable property. Then the empty space has property $Q$.*

**Proposition (1.7).** *Any $Q$-composable property is pairwise productive.*

It is immediate, of course, that if $P$ is any $Q$-composable property and $A$ and $B$ have property $P$, then $A \times B$ has property $Q$. It is only slightly less immediate that since $P$ is $Q$-composable, $A \times B$ in fact has property $P$.

It seems appropriate to include at this point a restatement of Theorem (2.6) of [3, p. 267]. In that paper it was assumed that "topological space" really meant "Hausdorff space" and what was proven there as Theorem (2.6) is, in our terminology here, the following

**Theorem (1.8).** *The property of being compact and Hausdorff is an $H$-composable property.*

Since the class $H$ contains the class $CR$, we immediately get the following

**Corollary (1.9).** *The property of being compact and Hausdorff is a $CR$-composable property.*

Later, we will get a result which will show that the property of being compact and Hausdorff is not $T$-composable. In proving our next theorem, we need the fact that if $P$ is any $Q$-composable property, then the one-point space has property $P$. With not much more effort, we can prove

**Lemma (1.10).** *Let $P$ be any $Q$-composable property. Then all finite discrete spaces have property $P$.*

**Proof.** It follows immediately from (1.5.1) and Proposition (1.7) that if $M$ is any power of 2, then the discrete space with $M$ elements has property $P$. Now let any $N$ be given and choose any $M > \max\{N, 2\}$ which is a power of 2. Let $X = \{x_1, x_2, x_3, \cdots, x_M\}$ be the discrete space with $M$ elements and let $\alpha$ and $\beta$ be the subspaces of $X \times X$ which are defined by

$$\alpha = \{(x_1, x_1), (x_1, x_2), \cdots, (x_1, x_N), (x_2, x_{N+1}), \cdots, (x_2, x_M)\},$$
$$\beta = \{(x_1, x_1), (x_1, x_3)\}.$$

Then both $\alpha$ and $\beta$ have property $P$ and, consequently, $\alpha \circ \beta$ has property $P$. But $\alpha \circ \beta = \{(x_1, x_1), x_1, x_2), \cdots, (x_1, x_N)\}$ is the discrete space with $N$ points.

**Proposition (1.11).** *Every $CR$-composable property is closed-hereditary.*

**Proof.** Let $P$ be any $CR$-composable property, let $X$ be any topological space with property $P$ and let $W$ be any closed subset of $X$. We must show that $W$ has property $P$. First of all, for each point $a$ in $X - W$, there exists a continuous function $f_a$ mapping $X$ into the closed unit interval such that $f_a(a) = 0$ and $f_a(x) = 1$ for $x \in W$. Now define a mapping $\alpha$ from $X$ into the product space $Y = \Pi\{X_a: a \in X - W\}$ by $(\alpha(x))_a = f_a(x)$. One easily verifies that $\alpha$ is a continuous function from $X$ into $Y$ and that $\alpha^{-1}(q) = W$ where $q$ is the point in $Y$ all of whose coordinates are 1. Since both $X$ and $Y$ are completely regular and Hausdorff, the free union $Z$ of $X$ and $Y$ is also. Now $\alpha^{-1}$ may be regarded as a relation on $Z$, that is, a subspace of $Z \times Z$ and, as such, it is homeomorphic to $X$. Specifically, since $\alpha$ is continuous, the mapping which takes a point $x \in X$ into the point $(\alpha(x), x)$ is a homomorphism from $X$ onto $\alpha^{-1}$ where $\alpha^{-1}$ has the topology inherited from $Z \times Z$. Thus, $\alpha^{-1}$ has the property $P$ and, by the previous lemma, the subspace $\{(q, q)\}$ of $Z \times Z$ also has property $P$. It follows that $\alpha^{-1} \circ \{(q, q)\}$ has property $P$ but

$$\alpha^{-1} \circ \{(q, q)\} = \{(q, x): x \in W\}$$

which is homeomorphic to $W$ so $W$ has property $P$.

**Corollary (1.12).** *Let $P$ be any $CR$-composable property. If $X$ and $Y$ have property $P$, then the free union of $X$ and $Y$ has property $P$.*

**Proof.** By the previous theorem, $P$ is closed-hereditary and by Proposition (1.7), it is pairwise productive. The conclusion follows from these facts and Proposition (1.4).

For $T$-composable properties, we have the following results which is analogous to Theorem (1.11):

**Proposition (1.13).** *Every $T$-composable property is hereditary.*

**Proof.** Let $P$ be any $T$-composable property. Let $X$ be any space with property $P$ and let $W$ be any subspace of $X$. We must show that $W$ has property $P$. First, choose a point $q \notin X$ and let $Y = (X - W) \cup \{q\}$. Define a mapping $\alpha$ from $X$ onto $Y$ by

$$\alpha(x) = x \quad \text{for } x \in X - W,$$
$$\alpha(x) = q \quad \text{for } x \in W.$$

Now we want to topologize $Y$ so that $\alpha$ is continuous. Any such topology (e.g., the quotient topology) will be sufficient for our needs. The proof from this point on is very similar to the proof of Theorem (1.11). Let $Z$ be the free union of $X$ and $Y$. Then $\alpha^{-1}$ is a subset of $Z \times Z$ and has property $P$ since it is homeomorphic to $X$. Thus $\alpha^{-1} \circ \{(q, q)\} = \{(q, w): w \in W\}$ has property $P$ which implies that $W$ has property $P$.

**Corollary (1.14).** *The property of being a compact Hausdorff space, although $H$-composable, is not $T$-composable.*

**Proposition (1.15).** *Let $P$ be any $CR$-composable property, let $X$ be any completely regular Hausdorff space and let $A$ and $B$ be any two subspaces of $X$. If both $A$ and $B$ have property $P$, then $A \cup B$ has property $P$.*

**Proof.** If $X$ does not have at least two points, the conclusion is immediate so assume $X$ has at least two distinct points $p$ and $q$. Define subsets $\alpha$ and $\beta$ of $X \times X$ by

$$\alpha = \{(p, x): x \in A\}, \qquad \beta = \{(q, x): x \in B\}.$$

Then both $\alpha$ and $\beta$ have property $P$ since they are homeomorphic to $A$ and $B$ respectively. Moreover, the topology induced on $\alpha \cup \beta$ by $X \times X$ is the topology of the free union of $\alpha$ and $\beta$. Thus, $\alpha \cup \beta$ has property $P$ by Corollary (1.12) and, since the two-point discrete space has property $P$, it follows that $(\alpha \cup \beta) \circ \{(p, p), (p, q)\}$ has the property $P$. However the latter is just the subspace $\{(p, x): x \in A \cup B\}$ which is homeomorphic to $A \cup B$.

As for intersections, we have

**Proposition (2.16).** *Let $Q$ be any bounding property and let $P$ be any $Q$-composable property. If $X$ has property $Q$ and $A$ and $B$ are any two subsets of $X$ with property $P$, then $A \cap B$ also has property $P$.*

**Proof.** Let $\alpha = \{(x, x): x \in A\}$ *and* $\beta = \{(x, x): x \in B\}$. Then $\alpha$ and $\beta$ are homeomorphic to $A$ and $B$ respectively and hence have property $P$. Thus $\alpha \circ \beta$ has property $P$ but

$$\alpha \circ \beta = \{(x, x): x \in A \cap B\}$$

is homeomorphic to $A \cap B$.

**Proposition (1.17).** *Let $P$ be any CR-composable property. Let $X$ be any space with property $P$ and let $Y$ be any completely regular Hausdorff space which is a continuous image of $X$. Then the space $Y$ also has property $P$.*

**Proof.** Let $\alpha$ be any continuous function from $X$ onto $Y$ and denote the free union of $X$ and $Y$ by $Z$. Certainly, $Z$ is completely regular and Hausdorff and the function $\alpha$, regarded as a subspace of $Z \times Z$, has property $P$ since it is homeomorphic to $X$. Choose any point $q \in Z$ and let $\beta = \{(q, x): x \in X\}$. Then $\beta$ also has property $P$ since, it, too, is homeomorphic to $X$. Consequently, $\alpha \circ \beta$ has property $P$. But $\alpha \circ \beta = \{(q, y): y \in Y\}$ which is homeomorphic to $Y$ and it follows that $Y$ has property $P$.

2. **Principal $Q$-composable properties.** At this point, we have all the facts we really need about $Q$-composable properties in order to carry out our proof of the homomorphism theorem. However, we include the following discussion since it will provide a bit more insight into the concept of $Q$-composable property.

For the initial portion of this discussion, we take $Q$ to be any topological property which is pairwise productive and hereditary. The class $\mathfrak{Q}$ of all $Q$-composable properties can be partially ordered by inclusion. Under this ordering, $\mathfrak{Q}$ has a smallest element $P_0$ and a largest element $P_1$. It is immediate that $P_1$ is simply $Q$ itself. We see by Lemma (1.10) that all finite discrete spaces belong to $P_0$. There may be more spaces but if all the spaces in $Q$ are Hausdorff, $P_0$ will consist precisely of all finite discrete spaces together with the empty space. This follows since the latter class of spaces is $Q$-composable whenever $Q \subset H$, the class of all Hausdorff spaces.

Now it is easily verified that any intersection of $Q$-composable properties is again a $Q$-composable property. Consequently, any subclass $C$ of $Q$ generates, in a natural way, a smallest $Q$-composable property $Q(C)$ such that each space in $C$ has property $Q(C)$. We make all this precise in the following

**Definition (2.1).** Let $Q$ be any topological property which is pairwise productive and hereditary and let $C$ be any subclass of $Q$. The intersection of all $Q$-composable properties which contain $C$ is a $Q$-composable property which we will denote by $Q(C)$ and refer to as the *$Q$-composable property which is generated by $C$.* Any $Q$-composable property which is generated by one space will be referred to as a *principal $Q$-composable property.*

**Theorem (2.2).** *The property of being a compact metric space is a principal*

*CR-composable property and is generated by any uncountable compact metric space.*

**Proof.** Let $M$ denote the property of being a compact metric space or, equivalently, let $M$ denote the class of all compact metric spaces. Certainly $M \subset CR$, the class of all completely regular Hausdorff spaces. Let $X \in CR$ and let $\alpha$ and $\beta$ be any two subspaces of $X \times X$ which are compact and metric. The sets $\{x: (x, y) \in \beta$ for some $y \in X\}$ and $\{y: (x, y) \in \alpha$ for some $x \in X\}$ are denoted respectively by $\mathfrak{D}(\beta)$ and $\mathfrak{R}(\alpha)$ and are referred to as the domain of $\beta$ and the range of $\alpha$. Now any Hausdorff space which is the continuous image of a compact metric space is itself a compact metric space. Thus, $\mathfrak{D}(\beta)$ and $\mathfrak{R}(\alpha)$ are metrizable since they are images of $\beta$ and $\alpha$ respectively under projection maps. Consequently, $\mathfrak{D}(\beta) \times \mathfrak{R}(\alpha)$ is metrizable which implies that $\alpha \circ \beta$ is metrizable since $\alpha \circ \beta \subset \mathfrak{D}(\beta) \times \mathfrak{R}(\alpha)$. It follows from Theorem (1.8) that $\alpha \circ \beta$ is compact. Thus, $\alpha \circ \beta$ has property $M$. That is to say, $M$ is $CR$-composable.

Now let $Y$ be any uncountable compact metric space. We must show that $M = CR(Y)$, the principal $CR$-composable property generated by $Y$. Since $Y$ has property $M$, it is immediate that $CR(Y) \subset M$. To get the inclusion in the other direction, let $P$ be any $CR$-composable property which contains the space $Y$. We need only show that $M \subset P$ since $CR(Y)$ is, by definition, the intersection of all such properties. By Corollary 1 of [2, p. 445], $Y$ contains a copy $C$ of the Cantor discontinuum. Since $Y$ has property $P$, the space $C$ must also have property $P$ in view of Proposition (1.11). But it is well known that any compact metric space is a continuous image of the Cantor discontinuum and it now follows from Proposition (1.16) that $M \subset P$.

**Corollary (2.3).** *Let $I$ denote the closed unit interval. The principal CR-composable property generated by $I$ is the property of being a compact metric space.*

One can also describe very concisely the principal $CR$-composable property which is generated by the open unit interval or, equivalently, by the space $R$ of real numbers.

**Theorem (2.4).** *The principal CR-composable property generated by $R$ is the property of being completely regular, Hausdorff and a countable union of compact metric spaces.*

**Proof.** Let $S$ denote the property of being completely regular, Hausdorff and a countable union of compact metric spaces and let $CR(R)$ denote the principal $CR$-composable property which is generated by the space $R$ of real numbers. First of all, we show that $S$ is $CR$-composable. Suppose $X$ is completely regular and Hausdorff and that $\alpha$ and $\beta$ are two relations on $X$ which have property $S$. Then $\alpha = \bigcup\{\alpha_n: n \in \omega\}$ and $\beta = \bigcup\{\beta_n: n \in \omega\}$ where $\omega$ denotes the set of natural numbers and each $\alpha_n$ and $\beta_n$ is a compact metric space. Then one easily verifies that

$$\alpha \circ \beta = \bigcup \{\alpha_m \circ \beta_n : m, n \in \omega\}.$$

Furthermore, each $\alpha_m \circ \beta_n$ is a compact metric space by Theorem (2.2). This verifies that $S$ is $CR$-composable and, since $R \in S$, it follows that $CR(R) \subset S$. To get the reverse inclusion, we need only show that if $P$ is any $CR$-composable property which contains $R$, then $S \subset P$. With this in mind, we let $P$ be such a property. Now the space of real numbers contains a closed subset $Y$ which is the free union of a countably infinite number of copies of the Cantor discontinuum. According to Proposition (1.11), the space $Y$ has property $P$. Moreover, since each compact metric space is a continuous image of the Cantor discontinuum, it readily follows that any completely regular Hausdorff space $Z$ which is a countable union of compact metric spaces is a continuous image of $Y$. Hence, by Proposition (1.16), any such space $Z$ must have property $P$. That is, $S \subset P$ and the conclusion follows.

As we observed in Theorem (2.2), the property of being a compact metric space is a principal $CR$-composable property. The property of being a compact Hausdorff space is, however, *not* a principal $CR$-composable property. One reason for this is that compact metric spaces cannot have arbitrarily large cardinalities while compact Hausdorff spaces can. We explore this a bit further. First of all, let us note that if $\lambda$ is any infinite cardinal number and $X$ is any space and $\alpha$ and $\beta$ are any two relations on $X$ such that card $\alpha \leq \lambda$ and card $\beta \leq \lambda$, then card $\alpha \circ \beta \leq \lambda$. This motivates the following

**Proposition (2.5).** *Let $Q$ be any topological property which is pairwise productive and hereditary and let $\lambda$ be any infinite cardinal number. Then*

$$Q_\lambda = \{X \in Q : \text{card } X \leq \lambda\}$$

*is a $Q$-composable property. Moreover, if the discrete space $D_\lambda$ with $\lambda$ elements belongs to $Q$, then $Q_\lambda$ is a principal $Q$-composable property which is generated by $D_\lambda$.*

**Proof.** As we observed above, $Q_\lambda$ is $Q$-composable. Suppose,in addition, that $D_\lambda \in Q$. Then, of course, $D_\lambda \in Q_\lambda$ and it follows that $Q(D_\lambda) \subset Q_\lambda$. On the other hand, if $P$ is any $Q$-composable property which contains $D_\lambda$, then $Q_\lambda \subset P$ since each space in $Q_\lambda$ is a continuous image of $D_\lambda$. This implies that $Q_\lambda \subset Q(D_\lambda)$ and it follows that $D_\lambda$ generates $Q_\lambda$.

This latter result is useful for showing that certain $Q$-composable properties are not principal.

**Proposition (2.6).** *Let $P$ be any $Q$-composable property and suppose that for* *each cardinal number* $\lambda$ *there exists a space* $X$ *with property* $P$ *such that* card $X > \lambda$. *Then* $P$ *is not a principal property.*

**Proof.** Let $X$ be any space with property $P$ and let $\lambda = \text{card } X$. Then $X \in Q_\lambda$ and, consequently, $Q(X) \subset Q_\lambda$. By hypothesis, there is a space $Y$ with property $P$ such that card $Y > \lambda$. Thus, $Y \notin Q(X)$. We have shown that no space with property $P$ generates $P$. Thus, $P$ is not principal.

**Corollary (2.7).** *The property of being a compact Hausdorff space is a CR-composable property which is not principal.*

We conclude this section with a few miscellaneous remarks. Among other things, we have shown that any $CR$-composable property is closed under the operation of taking finite unions and finite intersections. The analogous statements are not true for infinite unions and infinite intersections. The property of being compact and Hausdorff is a $CR$-composable property which is not closed under the operation of taking infinite unions. As for infinite intersections, let $P$ denote the property of being completely regular, Hausdorff and a countable union of compact metric spaces. As we noted previously, $P$ is a $CR$-composable property and is, in fact, the principal $CR$-composable property which is generated by the space $R$ of real numbers. Now for each rational number $r$, let $X_r = R - \{r\}$. Then each $X_r$ has property $P$ and $\bigcap \{X_r : r \text{ is rational}\}$ is just the space $J$ or irrational numbers. However $J$ does not have property $P$ since any compact subset of $J$ has empty interior and the assumption that $J$ is a countable union of compact metric spaces leads to the contradiction that $J$ is a first category space.

Our final remark of this section concerns local topological properties. Any topological property $P$ naturally gives rise to a second topological property which we refer to as *local* $P$. We say that a space $X$ has the property local $P$ if, for each point $x \in X$ and each open subset $G$ of $X$ containing $X$, there exists an open subset $H$ of $X$ and a subspace $A$ of $X$ which has property $P$ such that $x \in H \subset A \subset G$. It is quite natural to ask if local $P$ is $Q$-composable when $P$ is $Q$-composable. One does not need to search too far to find that the answer is no. In fact, local compactness is not $CR$-composable. Let $X$ denote the space of rational numbers and let $Y$ denote the countably infinite discrete space. Let $\alpha$ be any bijection from $X$ onto $Y$ and let $Z$ denote the free union of $X$ and $Y$. Then both $\alpha$ and $\alpha^{-1}$ can be regarded as subspaces of $Z \times Z$ and both are locally compact since they are discrete. However,

$$\alpha^{-1} \circ \alpha = \{(x, x) : x \text{ is a rational number}\}$$

is homeomorphic to the space of rational numbers and, consequently, is not locally compact.

3. **The homomorphism theorem and related results.** In this section, we prove a topological version of the theorem of Clifford and Miller [1] in which they determined all CM-endormorphisms of the semigroup of all binary relations on a set. For a somewhat analogous yet quite different development, see [4] and [5]. Before stating the theorem, some discussion is in order and we begin with

**Definition (3.1).** Let $X$ and $Y$ be two nonempty sets. Let $\mathfrak{A}_X$ be *any* semigroup of binary relations on $X$ and let $\mathfrak{B}_Y$ denote the semigroup of *all* binary relations on $Y$ where the multiplication in both cases is composition of relations. A homomorphism $\theta$ from $\mathfrak{A}_X$ into $\mathfrak{B}_Y$ is said to be a CM-homomorphism if it is nonconstant, takes symmetric relations into symmetric relations and is union preserving in the sense that, if $\alpha \in \mathfrak{A}_X$ and $\alpha = \bigcup\{\beta_a : a \in A\}$ where each $\beta_a \in \mathfrak{A}_X$, then $\theta(\alpha) = \bigcup\{\theta(\beta_a): a \in A\}$.

These are the homomorphisms we will be discussing. The semigroups we will be discussing are obtained as follows: let $P$ be any $CR$-composable property, let $X$ be any completely regular Hausdorff space and *let* $S_P[X]$ *denote the family of all binary relations on* $X$ *which, when regarded as a subspace of* $X \times X$, *have property* $P$. *Then* $S_P[X]$ *is a semigroup under composition.* Indeed, as we have mentioned earlier, this fact is what motivated the study of $Q$-composable properties in general. These are the semigroups we will be considering and in our first result of this section, we will determine all CM-homomorphisms from one such semigroup into another under suitable conditions on the spaces involved. Before stating the theorem, we need the following

**Definition (3.2).** Let $P$ be any topological property. A topological space $X$ is said to be $P$-admissible if every subset of both $X$ and $X \times X$ which has property $P$ is closed.

For example, if $P$ is compactness, then every Hausdorff space is $P$-admissible and any discrete space is $P$-admissible for any topological property $P$. Now we are in a position to state

**Theorem (3.3).** Let $P$ *and* $Q$ *be any two CR-composable properties. Let* $X$ *be a completely regular Hausdorff space which has property* $P$ *and is P-admissible and let* $Y$ *be any completely regular Hausdorff Q-admissible space. Let* $\pi$ *and* $\zeta$ *be two disjoint partial equivalences on* $Y$ (*i.e., relations which are both symmetric and transitive*) *such that* $\pi \neq \phi$ *and both* $\zeta$ *and* $\pi \cup \zeta$ *have property* $Q$. *Then let* $\mu$ *be any continuous function from the domain* $E$ *of* $\pi$ *onto* $X$ *such that*

$$\mu \circ \pi = E \times X. \tag{3.3.1}$$

*Define a mapping* $\theta$ *by*

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta \tag{3.3.2}$$

*for each* $\alpha \in S_P[X]$. *Then* $\theta$ *is a CM-homomorphism from* $S_P[X]$ *into* $S_Q[Y]$ *and every CM-homomorphism from* $S_P[X]$ *into* $S_Q[Y]$ *is obtained in precisely this manner.*

The proof of this result makes liberal use of the results of the first section of this paper and the theorem of Clifford and Miller which we now state in a form which is readily applicable here.

**Theorem (3.4) (Clifford and Miller).** *Let* $X$ *and* $Y$ *be any two nonempty sets. Let* $\mathfrak{A}_X$ *be any semigroup of binary relations on* $X$ *which contains all relations consisting of a single point and let* $\mathfrak{B}_Y$ *denote the semigroup of all binary relations on* $Y$. *Now let* $\pi$ *and* $\zeta$ *be any two disjoint partial equivalences on* $Y$ *such that* $\pi \neq \phi$ *and let* $\mu$ *be any function from the domain* $E$ *of* $\pi$ *onto* $X$ *such that*

$$\mu \circ \pi = E \times X. \tag{3.4.1}$$

*Define a mapping* $\theta$ *by*

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta. \tag{3.4.2}$$

*Then* $\theta$ *is a CM-homomorphism from* $\mathfrak{A}_X$ *into* $\mathfrak{B}_Y$ *and every CM-homomorphism from* $\mathfrak{A}_X$ *into* $\mathfrak{B}_Y$ *is obtained in precisely this manner.*

For the proof of this result, one should consult the proof of Theorem 3 of [1, p. 310]. It carries over with only minor modifications. And now we commence with the

**Proof of Theorem (3.3).** We note first that the one-point space has property $P$ because of Lemma (1.10). Thus, all relations on $X$ consisting of a single point belong to $S_P[X]$. That is, $S_P[X]$ satisfies the condition demanded of $\mathfrak{A}_X$ in the theorem of Clifford and Miller. Now let $\pi$ and $\zeta$ be disjoint partial equivalences on $Y$ such that $\pi \neq \phi$ and both $\zeta$ and $\pi \cup \zeta$ have property $Q$. Let $\mu$ be a continuous function from the domain $E$ of $\pi$ onto $X$ such that (3.3.1) is satisfied and define a mapping $\theta$ as in (3.3.2). It is immediate from Theorem (3.4) that $\theta$ is a CM-homomorphism from $S_P[X]$ into $\mathfrak{B}_Y$. We must show that $\theta$ actually maps $S_P[X]$ into $S_Q[Y]$, i.e., that $(\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$ has property $Q$ whenever $\alpha$ has property $P$. To aid us, we define a function $\eta$ from $E \times E$ into $X \times X$ by $\eta(x, y) = (\mu(x), \mu(y))$ and we observe that $\eta$ is continuous since $\mu$ is continuous. Moreover,

$$\eta^{-1}(\alpha) = \mu^{-1} \circ \alpha \circ \mu \tag{3.3.3}$$

for each subset $\alpha$ of $X \times X$. So suppose $\alpha \subset X \times X$ has property $P$. Since $X$ is $P$-admissible, $\alpha$ is closed and hence by (3.3.3), $\mu^{-1} \circ \alpha \circ \mu$ is closed in $E \times E$. This implies that

$$\pi \cap \mu^{-1} \circ \alpha \circ \mu \text{ is a closed subset of } \pi \tag{3.3.4}$$

so there exists a closed subset $H$ of $Y \times Y$ such that

$$\pi \cap \mu^{-1} \circ \alpha \circ \mu = H \cap \pi. \tag{3.3.5}$$

Now $\pi \cup \zeta$ has property $Q$ so $\pi \cup \zeta$ is closed in $Y \times Y$ since $Y$ is $Q$-admissible. Then we have

$$(\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta = (H \cap \pi) \cup \zeta = (H \cup \zeta) \cap (\pi \cup \zeta).$$

But $\zeta$ has property $Q$ and must be closed in $Y \times Y$ since $Y$ is $Q$-admissible. Thus

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta$$

is closed in $Y \times Y$ since the three sets $H, \zeta$, and $\pi \cup \zeta$ are all closed in $Y \times Y$. Then $\theta(\alpha)$ is also closed in $\pi \cup \zeta$ and it now follows from Proposition (1.11) that $\theta(\alpha)$ has property $Q$, that is, $\theta$ is indeed a CM-homomorphism from $S_P[X]$ into $S_Q[Y]$.

Now we prove the converse. Let $\theta$ be any CM-homomorphism from $S_P[X]$ into $S_Q[Y]$. From Theorem (3.4), we immediately conclude the existence of partial equivalences $\pi$ and $\zeta$ with $\pi \neq \phi$ and a mapping $\mu$ from $E$ onto $X$ such that both (3.3.1) and (3.3.2) are satisfied. Our task is to show that $\zeta$ and $\pi \cup \zeta$ both have property $Q$ and that $\mu$ is continuous. Since the empty space has property $P$ and $\theta(\phi) = \zeta$, it follows that $\zeta$ must have property $Q$. Moreover, since $X$ has property $P$, Proposition (1.7) assures us that $X \times X$ has property $P$. Consequently,

$$\theta(X \times X) = (\pi \cap \mu^{-1} \circ X \times X \circ \mu) \cup \zeta = \pi \cup \zeta$$

has property $Q$. We have yet to show that $\mu$ is continuous and, as a first approximation, we prove that

$$\mu^{-1}(\mathfrak{D}(\alpha)) = E \cap \mathfrak{D}(\theta(\alpha)) \tag{3.3.5}$$

where $\mathfrak{D}(\alpha)$ and $\mathfrak{D}(\theta(\alpha))$ denote the domains of $\alpha$ and $\theta(\alpha)$ respectively and $\alpha \in S_P[X]$. Suppose $a \in \mu^{-1}(\mathfrak{D}(\alpha))$. Then $a \in E$ and $(\mu(a), b) \in \alpha$ for some $b \in X$. The point $(a, b)$ belongs to $E \times X$ which, according to (3.4.1), is $\mu \circ \pi$. Thus, there exists a point $c \in Y$ such that $(a, c) \in \pi$ and $(c, b) \in \mu$. Then $(a, \mu(a)) \in \mu$, $(\mu(a), b) \in \alpha$ and $(b, c) \in \mu^{-1}$ all of which imply that $(a, c) \in \pi \cap \mu^{-1} \circ \alpha \circ \mu$. This places $a \in \mathfrak{D}(\theta(\alpha))$ because of (3.3.2).

Suppose, on the other hand, that $a \in E \cap \mathfrak{D}(\theta(\alpha))$. Since $\pi$ and $\zeta$ are disjoint partial equivalences, their domains are disjoint. Thus, $a \in E$ implies $a \notin \mathfrak{D}(\zeta)$. This fact and (3.3.2) together imply that $(a, b) \in \pi \cap \mu^{-1} \circ \alpha \circ \mu$ for some $b \in Y$ and it readily follows that $a \in \mu^{-1}(\mathfrak{D}(\alpha))$. Now we use (3.3.5) to show that $\mu$ is continuous. Let $A$ be any closed subset of $X$. Since $X$ has property $P$, it follows from Proposition (1.11) that $A$ has property $P$. Choose any point $p \in X$

and let $\alpha = \{(x, p): x \in A\}$. Then $\alpha$ has property $P$ since it is homeomorphic to $A$ and, by (3.3.5), we have

$$\mu^{-1}(A) = \mu^{-1}(\mathfrak{D}(\alpha)) = E \cap \mathfrak{D}(\theta(\alpha)).$$

But $\theta(\alpha)$ has property $Q$ and since $\mathfrak{D}(\theta(\alpha))$ is a continuous image of $\theta(\alpha)$ under the projection mapping, it too must have property $Q$ because of Proposition (1.17). This implies that $\mathfrak{D}(\theta(\alpha))$ is closed in $Y$ since $Y$ is $Q$-admissible and this in turn implies that $E \cap \mathfrak{D}(\theta(\alpha))$ is closed in $E$. Hence, $\mu$ is continuous and the proof is complete.

One can show that any CM-homomorphism from $S_P[X]$ into $S_Q[Y]$ is injective so that, in fact, any such homomorphism from $S_P[X]$ into $S_Q[Y]$ is actually an embedding. We do not give the proof since it differs very little from the proof of Theorem (3.4) of [5]. For some related results, see [6]. All this motivates the following

**Definition (3.5).** We say that $S_P[X]$ can be CM-embedded in $S_Q[Y]$ if there exists a CM-homomorphism from $S_P[X]$ into $S_Q[Y]$.

Our next result gives a necessary and sufficient condition that $S_P[X]$ can be CM-embedded in $S_Q[Y]$ when the spaces $X$ and $Y$ satisfy the conditions of Theorem (3.3).

**Theorem (3.5).** *Let $P$ and $Q$ be any two CR-composable properties. Let $X$ be a completely regular Hausdorff space which has property $P$ and is $P$-admissible and let $Y$ be any completely regular Hausdorff $Q$-admissible space. Then $S_P[X]$ can be CM-embedded in $S_Q[Y]$ if and only if some nonempty subspace of $Y$ with property $Q$ maps continuously onto $X$.*

**Proof.** First suppose that some nonempty subspace $E$ of $Y$ has property $Q$ and maps continuously onto $X$ with some function $\mu$. Take $\pi = E \times E$ and $\zeta = \phi$. Since any $CR$-composable property is pairwise productive and the empty space has property $Q$, it follows that both $\zeta$ and $\pi \cup \zeta$ have property $Q$. Then $\theta$ defined as in (3.3.2) is a CM-homomorphism from $S_P[X]$ into $S_Q[Y]$.

Now suppose that $S_P[X]$ can be CM-embedded in $S_Q[Y]$, i.e., that there exists a CM-homomorphism $\theta$ from $S_P[X]$ into $S_Q[Y]$. Then there exist $\pi, \zeta$ and $\mu$ satisfying the conditions given in Theorem (3.3). Since $\pi \neq \phi$, we can choose a point $p$ in the range $\mathfrak{R}(\pi)$ of $\pi$. Since $\pi$ and $\zeta$ are disjoint partial equivalences, their ranges are also disjoint and it follows that $p \notin \mathfrak{R}(\zeta)$. This implies that $\langle p \rangle \circ \zeta = \phi$ where $\langle p \rangle$ is used to denote the relation $\{(p, p)\}$. Now since $Q$ is $CR$-composable, all one-point spaces have property $Q$ and since $\pi \cup \zeta$ has property $Q$, we conclude that

$$\langle p \rangle \circ (\pi \cup \zeta) = (\langle p \rangle \circ \pi) \cup (\langle p \rangle \circ \zeta) = \langle p \rangle \circ \pi$$

has property $Q$. But $\langle p \rangle \circ \pi$ maps continuously onto $\mathfrak{D}(\langle p \rangle \circ \pi)$ with the projection mapping so it follows from Proposition (1.17) that $\mathfrak{D}(\langle p\rangle \circ \pi)$ also has property $Q$. But

$$\mathfrak{D}(\langle p\rangle \circ \pi) = \{y \in Y\colon (y, p) \in \pi\},$$

so in order to complete the proof, we need only show that the latter set maps continuously onto $X$. Evidently, $\mu$ maps the set continuously into $X$ since it is a subset of $E$. To see that $\mu$ actually maps it onto $X$, let any $x \in X$ be given. Then $(p, x) \in E \times X$ which, by (3.3.1), is $\mu \circ \pi$. Hence there exists a point $a \in Y$ such that $(p, a) \in \pi$ and $(a, x) \in \mu$. Since $\pi$ is symmetric, $(a, p) \in \pi$ which implies that $a \in \mathfrak{D}(\langle p\rangle \circ \pi)$. This completes the proof since $\mu(a) = x$.

**Corollary (3.6).** *Let $P$ and $Q$ be any two CR-composable properties. Let $X$ be a completely regular Hausdorff space which has property $P$ and is $P$-admissible and let $Y$ be any completely regular Hausdorff $Q$-admissible space. If $S_P[X]$ can be CM-embedded in $S_Q[Y]$, then $X$ must have property $Q$.*

**Proof.** This is an immediate consequence of the previous theorem and Proposition (1.17).

**4. Some applications of the previous results.** Let $K$ denote the property of being compact and Hausdorff. By Corollary (1.9), $K$ is $CR$-composable and, since any Hausdorff space is $K$-admissible, the following two results are immediate consequences of Theorems (3.3) and (3.5) respectively.

**Theorem (4.1).** *Let $X$ be a compact Hausdorff space, $Y$ a completely regular Hausdorff space and let $S_K[X]$ and $S_K[Y]$ denote the semigroups of all compact binary relations on $X$ and $Y$ respectively. Let $\pi$ and $\zeta$ be disjoint partial equivalences on $Y$ such that $\pi \neq \phi$ and both $\zeta$ and $\pi \cup \zeta$ are compact. Let $\mu$ be any continuous function from the domain $E$ of $\pi$ onto $X$ such that*

$$\mu \circ \pi = E \times X \tag{4.1.1}$$

*and define a mapping $\theta$ by*

$$\theta(\alpha) = (\pi \cap \mu^{-1} \circ \alpha \circ \mu) \cup \zeta \tag{4.1.2}$$

*for each $\alpha \in S_K[X]$. Then $\theta$ is a CM-homomorphism from $S_K[X]$ into $S_K[Y]$ and every CM-homomorphism from $S_K[X]$ into $S_K[Y]$ is obtained in precisely this manner.*

**Theorem (4.2).** *Again, let $X$ be a compact Hausdorff space and let $Y$ be any completely regular Hausdorff space. Then $S_K[X]$ can be CM-embedded in $S_K[Y]$ if and only if some compact subspace of $Y$ maps continuously onto $X$.*

We use this result to prove

**Theorem (4.3).** *Let $X$ be a compact Hausdorff space and let $Y$ be an uncountable complete separable metric space. Then $S_K[X]$ can be CM-embedded in $S_K[Y]$ if and only if $X$ is a metric space.*

**Proof. (Sufficiency).** Suppose $X$ is metric as well as compact. By Corollary 1 of [2, p. 445], $Y$ contains a copy of the Cantor discontinuum which, by a well-known result of Alexandroff, maps continuously onto every compact metric space. Thus, by Theorem (4.2), $S_K[X]$ can be CM-embedded in $S_K[Y]$.

**(Necessity).** Now suppose that $S_K[X]$ can be CM-embedded in $S_K[Y]$. Then by Theorem (4.2), some compact subspace $W$ of $Y$ maps continuously onto $X$. Since $Y$ has a countable base, $W$ does also and since the weight of any compact Hausdorff space cannot exceed the weight of any space which maps continuously onto it, the space $X$ must have a countable base. Since it is also compact and Hausdorff, it is metrizable.

We remark that one can take any $CR$-composable property $P$ such that any space with property $P$ is also compact, and use Theorems (3.3) and (3.5) to immediately get results analogous to Theorems (4.1) and (4.2). Examples of such properties are the property of being compact and metrizable and also the property of being compact and countable.

In order to illustrate the sort of information we can get from Corollary (3.6), let $P$ denote the property of being compact and Hausdorff and let $M$ denote the property of being compact and metrizable. It follows from Corollary (3.6) that if $X$ is any compact Hausdorff space which is not metrizable, then $S_P[X]$ cannot be CM-embedded in $S_M[Y]$ for any completely regular Hausdorff space $Y$. However, if $X$ is metrizable, Theorem (3.5) assures us that there are many spaces $Y$ such that $S_P[X]$ can be CM-embedded in $S_M[Y]$. One need only choose a completely regular Hausdorff space containing a compact subspace which maps continuously onto $X$. In fact, it follows from Theorem (4.3) that any uncountable complete separable metric space will suffice.

We conclude with one more observation. It is immediate that the property $CR$ of being completely regular and Hausdorff is $CR$-composable and, of course, any discrete space is $CR$-admissible. So if, in Theorem (3.3), we take $X = Y$ to be discrete and $P = Q = CR$, then $S_{CR}[X] = S_{CR}[Y]$ is just the semigroup of *all* binary relations on $X$ and we have Theorem 3 of [1, p. 310].

**Added in proof.** We have shown that any $CR$-composable property is (1) closed under continuous images, (2) pairwise productive, (3) closed-hereditary and (4) closed under finite intersections. J. Lawson and B. Madison have pointed out to the authors that if the property $P \subset CR$, and it contains the two-point discrete space and it satisfies (1), (2) and (3) then $P$ is $CR$-composable. This follows from the fact that

$$\alpha \circ \beta = \Pi((\beta \times \alpha) \cap (X \times \Delta(X \times X) \times X))$$

where $\alpha, \beta \subset X \times X$, $\Delta(X \times X)$ is the diagonal of $X \times X$ and $\Pi$ is the mapping which sends $(a, b, c, d)$ into $(a, d)$. One can modify their argument slightly and also get the result that if $P \subset CR$ and it contains the two-point discrete space and it satisfies (1), (2) and (4), then $P$ is $CR$-composable. The crucial observation here is that

$$\alpha \circ \beta = \Pi((\beta \times \alpha) \cap (\mathfrak{D}(\beta) \times \Delta(\mathfrak{R}(\beta) \times \mathfrak{R}(\beta)) \times \mathfrak{R}(\alpha))).$$

It follows from all this that if $P \subset CR$ and contains the two-point discrete space and $P$ is both pairwise productive and closed under continuous images, then $P$ is closed under finite intersections if and only if it is closed-hereditary.

## REFERENCES

1. A. H. Clifford and D. D. Miller, *Union and symmetry preserving endomorphisms of the semigroup of all binary relations on a set*, Czechoslovak Math. J. 20 (95) (1970), 303–314. MR 41 #7003.

2. K. Kuratowski, *Topologie*. Vol. 1, PWN, Warsaw, 1958; English transl., Academic Press, New York; PWN, Warsaw, 1966. MR 19, 873; MR 36 #839.

3. K. D. Magill, Jr., *Composable topological properties and semigroups of relations*, J. Austral. Math. Soc. 11 (1970), 265–275. MR 42 #5209.

4. K. D. Magill, Jr. and S. Yamamuro, *Union and symmetry preserving homomorphisms of semigroups of closed relations*, Research Announcement, Semigroup Forum 3 (1971), 168–172.

5. ———, *Union and symmetry preserving homomorphisms of semigroups of closed relations*, Proc. London Math. Soc. (3) 24 (1972), 59–78.

6. D. B. McAlister, *Homomorphisms of semigroups of binary relations*, Semigroup Forum 3 (1971), 185–188.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601

DEPARTMENT OF MATHEMATICS, STATE UNIVERSITY OF NEW YORK AT BUFFALO, AMHERST, NEW YORK 14226